Big Ideas Math: Algebra 2

Chapter Summary

Chapter 9: Trigonometric Ratios and Functions

Learning Goals

Evaluate trigonometric functions of acute angles.

Find unknown side lengths and angle measures of right triangles.

Use trigonometric functions to solve real-life problems.

Draw angles in standard position.

Find coterminal angles.

Use radian measure.

Evaluate trigonometric functions of any angle.

Interpret and use frequency.

Write trigonometric functions.

Use technology to find trigonometric models.

Use trigonometric identities to evaluate trigonometric functions and simplify trigonometric expressions.

Verify trigonometric identities.

Use sum and difference formulas to evaluate and simplify trigonometric expressions.

Graph secant and cosecant functions.

Use sum and difference formulas to solve trigonometric equations and rewrite real-life formulas.

Find and use reference angles to evaluate trigonometric functions.

Explore characteristics of sine and cosine functions.

Stretch and shrink graphs of sine and cosine functions.

Graph tangent and cotangent functions.

Translate graphs of sine and cosine functions.

Reflect graphs of sine and cosine functions.

Explore characteristics of tangent and cotangent functions.

Standards

Common Core: HSA-CED.A.2, HSF-IF.C.7e, HSF-BF.A.1a, HSF-BF.B.3, HSF-TF.A.1, HSF-TF.A.2, HSF-TF.B.5, HSF-TF.C.8, HSF-TF.C.9

Game

- Getting Triggy with Triangles

This is available online in the *Game Closet* at www.bigideasmath.com.

Essential Questions

How can you find a trigonometric function of an acute angle θ?

How can you find the measure of an angle in radians?

How can you use the unit circle to define the trigonometric functions of any angle?

What are the characteristics of the graphs of the sine and cosine functions?

What are the characteristics of the graph of the tangent function?

What are the characteristics of the real-life problems that can be modeled by trigonometric functions?

How can you verify a trigonometric identity?

How can you evaluate trigonometric functions of the sum or difference of two angles?

Core Vocabulary

Ratios of a right triangle's side lengths are used to define the six trigonometric functions: ***sine***, ***cosine***, ***tangent***, ***cosecant***, ***secant***, and ***cotangent***.

The fixed ray of an angle in standard position in a coordinate plane is called the ***initial side***.

A ray of an angle in standard position that has been rotated about the vertex in a coordinate plane is called the ***terminal side***.

An angle is in ***standard position*** when its vertex is at the origin and its initial side lies on the positive x-axis.

Two angles whose terminal sides coincide are ***coterminal***.

For a circle with radius r, the measure of an angle in standard position whose terminal side intercepts an arc of length r is one ***radian***.

A ***sector*** is a region of a circle that is bounded by two radii and an arc of the circle.

The ***central angle*** θ of a sector is the angle formed by the two radii.

The circle $x^2 + y^2 = 1$, which has center (0, 0) and radius 1, is called the ***unit circle***.

An angle in standard position whose terminal side lies on an axis is called a ***quadrantal angle***.

The ***reference angle*** for θ is the acute angle θ' formed by the terminal side of θ and the x-axis.

The ***amplitude*** is one-half the difference of the maximum value and the minimum value of the graph of a trigonometric function.

A function whose graph has a repeating pattern is ***periodic function***.

The shortest repeating portion of the graph of a periodic function is called a ***cycle***.

The horizontal length of each cycle of a periodic function is called the ***period***.

A horizontal translation of a periodic function is called a ***phase shift***.

The ***midline*** of a graph is the horizontal line $y = k$ in which the graph of a periodic function oscillates.

The number of cycles per unit of time, which is the reciprocal of the period, is called the ***frequency***.

Graphs of sine and cosine functions are called ***sinusoids***.

A trigonometric equation that is true for all values of the variable for which both sides of the equation are defined is called a ***trigonometric identity***.

Core Concept

Right Triangle Definitions of Trigonometric Functions

Let θ be an acute angle of a right triangle. The six trigonometric functions of θ are defined as shown.

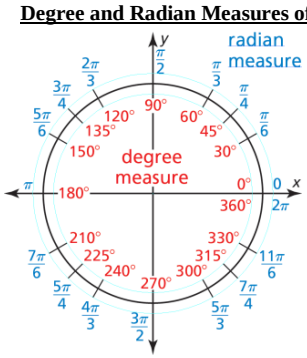
$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ $\quad\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$ $\quad\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$

$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$ $\quad\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$ $\quad\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$

The abbreviations *opp.*, *adj.*, and *hyp.* are often used to represent the side lengths of the right triangle. Note that the ratios in the second row are reciprocals of the ratios in the first row.

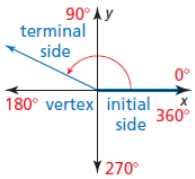
$\csc\theta = \frac{1}{\sin\theta}$ $\quad\sec\theta = \frac{1}{\cos\theta}$ $\quad\cot\theta = \frac{1}{\tan\theta}$

Degree and Radian Measures of Special Angles



- The diagram shows equivalent degree and radian measures for special angles from 0° to 360° (0 radians to 2π radians).
- You may find it helpful to memorize the equivalent degree and radian measures of special angles in the first quadrant and for $90° = \frac{\pi}{2}$ radians. All other special angles shown are multiples of these angles.

Angles in Standard Position



- In a coordinate plane, an angle can be formed by fixing one ray, called the initial side, and rotating the other ray, called the terminal side, about the vertex.
- An angle is in standard position when its vertex is at the origin and its initial side lies on the positive x-axis.

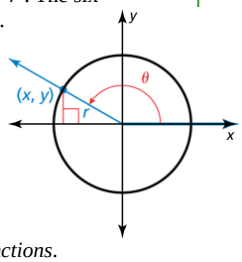
General Definitions of Trigonometric Functions

Let θ be an angle in standard position, and let (x, y) be the point where the terminal side of θ intersects the circle $x^2 + y^2 = r^2$. The six trigonometric functions of θ are defined as shown.

$\sin\theta = \frac{y}{r}$ $\quad\csc\theta = \frac{r}{y}, y \neq 0$

$\cos\theta = \frac{x}{r}$ $\quad\sec\theta = \frac{r}{x}, x \neq 0$

$\tan\theta = \frac{y}{x}, x \neq 0$ $\quad\cot\theta = \frac{x}{y}, y \neq 0$

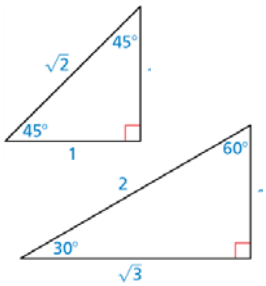


These functions are sometimes called *circular functions*.

Trigonometric Values for Special Angles

The table gives the values of the six trigonometric functions for the angles 30°, 45°, and 60°. You can obtain these values from the triangles shown.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$



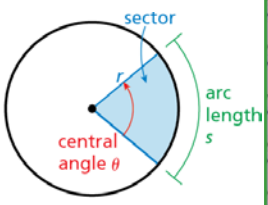
Converting Between Degrees and Radians

Degrees to radians
Multiply degree measure by $\frac{\pi \text{ radians}}{180°}$.

Radians to degrees
Multiply radian measure by $\frac{180°}{\pi \text{ radians}}$.

Arc Length and Area of a Sector

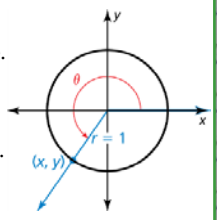
The arc length s and area A of a sector with radius r and central angle θ (measured in radians) are as follows.



Arc length: $s = r\theta$
Area: $A = \frac{1}{2}r^2\theta$

The Unit Circle

The circle $x^2 + y^2 = 1$, which has center (0, 0) and radius 1, is called the unit circle. The values of $\sin\theta$ and $\cos\theta$ are simply the y-coordinate and x-coordinate, respectively, of the point where the terminal side of θ intersects the unit circle.



$\sin\theta = \frac{y}{r} = \frac{y}{1} = y$
$\cos\theta = \frac{x}{r} = \frac{x}{1} = x$

Characteristics of $y = \sin x$ and $y = \cos x$

- The domain of each function is all real numbers.
- The range of each function is $-1 \le y \le 1$. So, the minimum value of each function is -1 and the maximum value is 1.
- The amplitude of the graph of each function is one-half of the difference of the maximum value and the minimum value, or $\frac{1}{2}[1 - (-1)] = 1$.
- Each function is periodic, which means that its graph has a repeating pattern. The shortest repeating portion of the graph is called a cycle. The horizontal length of each cycle is called the period. Each graph shown above has a period of 2π.
- The x-intercepts for $y = \sin x$ occur when $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.
- The x-intercepts for $y = \cos x$ occur when $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$

Fundamental Trigonometric Identities

Reciprocal Identities
$\csc\theta = \frac{1}{\sin\theta}$ $\quad\sec\theta = \frac{1}{\cos\theta}$
$\cot\theta = \frac{1}{\tan\theta}$

Tangent and Cotangent Identities
$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\quad\cot\theta = \frac{\cos\theta}{\sin\theta}$

Pythagorean Identities
$\sin^2\theta + \cos^2\theta = 1$ $\quad 1 + \tan^2\theta = \sec^2\theta$
$1 + \cot^2\theta = \csc^2\theta$

Cofunction Identities
$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ $\quad\cos = \left(\frac{\pi}{2} - \theta\right) = \sin\theta$
$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$

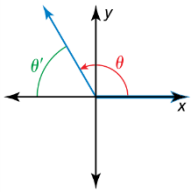
Negative Angle Identities
$\sin(-\theta) = -\sin\theta$ $\quad\cos(-\theta) = \cos\theta$
$\tan(-\theta) = -\tan\theta$

Core Concept

Reference Angle Relationships

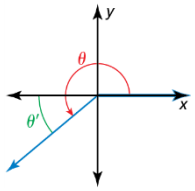
Let θ be an angle in standard position. The reference angle for θ is the acute angle θ' formed by the terminal side of θ and the x-axis. The relationship between θ and θ' is shown below for nonquadrantal angles θ such that $90° < \theta < 360°$ or, in radians, $\frac{\pi}{2} < \theta < 2\pi$.

Quadrant II



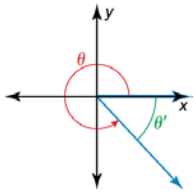
Degrees: $\theta' = 180° - \theta$
Radians: $\theta' = \pi - \theta$

Quadrant III



Degrees: $\theta' = \theta - 180°$
Radians: $\theta' = \theta - \pi$

Quadrant IV



Degrees: $\theta' = 360° - \theta$
Radians: $\theta' = 2\pi - \theta$

Evaluating Trigonometric Functions

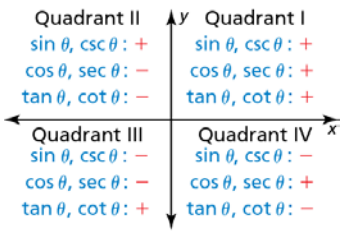
Use these steps to evaluate a trigonometric function for any angle θ:

Step 1 Find the reference angle θ'.

Step 2 Evaluate the trigonometric function for θ'.

Step 3 Determine the sign of the trigonometric function value from the quadrant in which θ lies.

Signs of Function Values



Amplitude and Period

The amplitude and period of the graphs of $y = a \sin bx$ and $y = a \cos bx$, where a and b are nonzero real numbers, are as follows:

Amplitude $= |a|$ Period $= \frac{2\pi}{|b|}$

Sum and Difference Formulas

Sum Formulas

$\sin(a + b) = \sin a \cos b + \cos a \sin b$

$\cos(a + b) = \cos a \cos b - \sin a \sin b$

$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$

Difference Formulas

$\sin(a - b) = \sin a \cos b - \cos a \sin b$

$\cos(a - b) = \cos a \cos b + \sin a \sin b$

$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$

Characteristics of $y = \tan x$ and $y = \cot x$

The functions $y = \tan x$ and $y = \cot x$ have the following characteristics.

- The domain of $y = \tan x$ is all real numbers except odd multiples of $\frac{\pi}{2}$. At these x-values, the graph has vertical asymptotes.
- The domain of $y = \cot x$ is all real numbers except multiples of π. At these x-values, the graph has vertical asymptotes.
- The range of each function is all real numbers. So, the functions do not have maximum or minimum values, and the graphs do not have an amplitude.
- The period of each graph is π.
- The x-intercepts for $y = \tan x$ occur when $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.
- The x-intercepts for $y = \cot x$ occur when $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$.

Period and Vertical Asymptotes of $y = a \tan bx$ and $y = a \cot bx$

The period and vertical asymptotes of the graphs of $y = a \tan bx$ and $y = a \cot bx$, where a and b are nonzero real numbers, are as follows.

- The period of the graph of each function is $\frac{\pi}{|b|}$.
- The vertical asymptotes for $y = a \tan bx$ are at odd multiples of $\frac{\pi}{2|b|}$.
- The vertical asymptotes for $y = a \cot bx$ are at multiples of $\frac{\pi}{|b|}$.

Characteristics of $y = \sec x$ and $y = \csc x$

The functions $y = \sec x$ and $y = \csc x$ have the following characteristics.

- The domain of $y = \sec x$ is all real numbers except odd multiples of $\frac{\pi}{2}$. At these x-values, the graph has vertical asymptotes.
- The domain of $y = \csc x$ is all real numbers except multiples of π. At these x-values, the graph has vertical asymptotes.
- The range of each function is $y \le -1$ and $y \ge 1$. So, the graphs do not have an amplitude.
- The period of each graph is 2π.

Graphing $y = a \sin b(x - h) + k$ and $y = a \cos b(x - h) + k$

To graph $y = a \sin b(x - h) + k$ or $y = a \cos b(x - h) + k$ where $a > 0$ and $b > 0$, follow these steps:

Step 1 Identify the amplitude a, the period $\frac{2\pi}{b}$, the horizontal shift h, and the vertical shift k of the graph.
Step 2 Draw the horizontal line $y = k$, called the midline of the graph.
Step 3 Find the five key points by translating the key points of $y = a \sin bx$ or $y = a \cos bx$ horizontally h units and vertically k units.
Step 4 Draw the graph through the five translated key points.

Additional Review

- Frequency, *p. 506*
- Writing Trigonometric Functions, p. 507
- Using Technology to Find Trigonometric Models, *p. 509*
- Trigonometric Equations and Real-Life Formulas, *p. 522*

What's the Point?

The STEM Videos available online show ways to use mathematics in real-life situations.

The Chapter 9: Parasailing to Great Heights STEM Video is available online at www.bigideasmath.com.